Geometric applications of calculus

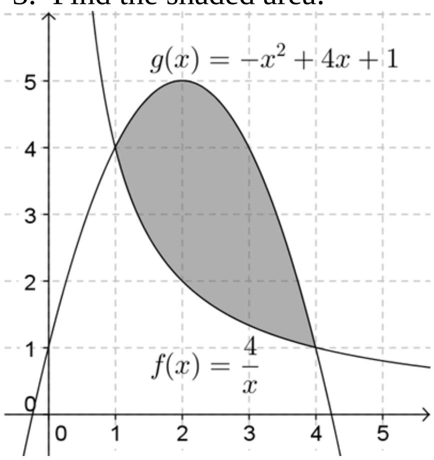
Tangent lines:

1. Find the equation of tangent line to the function $y = x^2 - 2x$ at the point where $x = 4$

2. Find the equation of the tangent line to the function $y = \ln(x)$ at the point where $x = 5$

Area questions can be asked in several ways:

3. Find the shaded area:



4. Find the area between the functions
$f(x) = x^2 - x - 1$ and $g(x) = 2x + 3$

5. Find the area between the functions $f(x) = \sqrt{x}$ and $g(x) = x^2 + 2$ with bounds $x = 1$ and $x = 3$

6. Find the area between the functions $f(x) = x^2 - x - 1$ and $g(x) = 2x + 3$ with bounds $x = -2$ and $x = 2$

Answers on next page.

1. Find the equation of tangent line to the function $y = x^2 - 2x$ at the point where $x = 4$

To find the y-coordinate of the point, plug in to the original function: $y = 4^2 - 4 \cdot 2 = 8$
To find the slope of the line, take the derivative and plug in 4:

$$y' = 2x - 2$$

$$m = 2 \cdot 4 - 2 = 6$$

Substitute the point and slope into the line function and solve for *b*:

$$8 = 6 \cdot 4 + b$$

$$8 - 24 = b$$

$$-16 = b$$

Finally, write the equation of the line:

$$y = 6x - 16$$

2. Find the equation of the tangent line to the function $y = \ln(x)$ at the point where $x = 5$

To find the y-coordinate of the point, plug in to the original function: $y = \ln(5) \approx 1.6094$
To find the slope of the line, take the derivative and plug in 5:

$$y' = \frac{1}{x}$$

$$m = \frac{1}{5}$$

Substitute the point and slope into the line function and solve for *b*:

$$1.6094 = \frac{1}{5} \cdot 5 + b$$

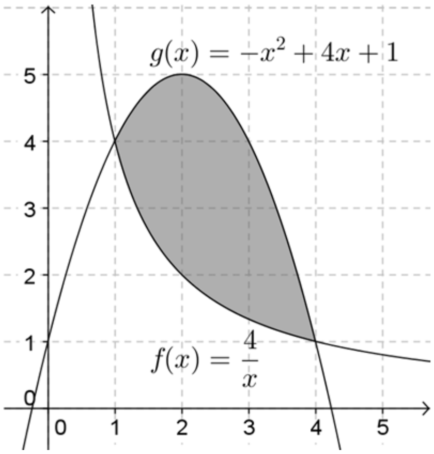
$$1.6094 - 1 = b$$

$$.6094 = b$$

Finally, write the equation of the line:

$$y = \frac{1}{5}x + 0.6094$$

3. Find the shaded area:



Look at the graph to find the x-interval endpoints, and set up the integral:

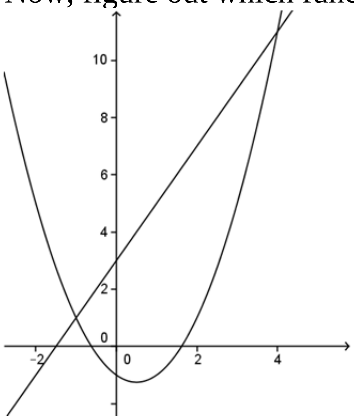
$$\int_1^4 -x^2 + 4x + 1 - \frac{4}{x}\,dx$$

Then find the anti-derivative in plug in the endpoints to find the area:

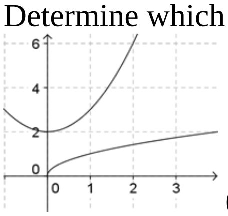
$$\int_1^4 -x^2 + 4x + 1 - 4\cdot\frac{1}{x}\,dx = -\frac{x^3}{3} + 4\cdot\frac{x^2}{2} + x - 4\ln(x)\Big|_1^4 = -\frac{x^3}{3} + 2x^2 + x - 4\ln(x)\Big|_1^4$$

$$= -\frac{4^3}{3} + 2\cdot 4^2 + 4 - 4\ln(4) - \left(-\frac{1^3}{3} + 2\cdot 1^2 + 1 - 4\ln(1)\right)$$

$$\approx -\frac{64}{3} + 32 + 4 - 4\cdot 1.386 - \left(-\frac{1}{3} + 2 + 1 - 0\right) \approx 6.4548$$

4. Find the area between the functions
$f(x)=x^2-x-1$ and $g(x)=2x+3$
First find where the functions intersect:
$x^2-x-1=2x+3$
$x^2-3x-4=0$
$(x-4)(x+1)=$
$x=4,-1$
Now, figure out which function is higher in the interval [-1,4]


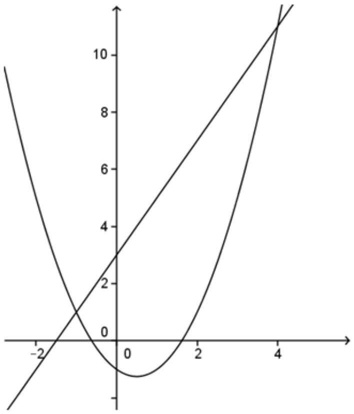
Set up the integral and integrate:
$$\int_{-1}^{4} 2x+3-(x^2-x-1)dx=\int_{-1}^{4} 2x+3+x^2+x+1dx=\int_{-1}^{4} x^2+3x+4dx$$
$$=\frac{x^3}{3}+3\cdot\frac{x^2}{2}+4x\bigg|_{-1}^{4}=\frac{4^3}{3}+3\cdot\frac{4^2}{2}+4\cdot4-\left(\frac{(-1)^3}{3}+3\cdot\frac{(-1)^2}{2}+4\cdot(-1)\right)=\frac{125}{6}\approx 20.8333$$

5. Find the area between the functions $f(x)=\sqrt{x}$ and $g(x)=x^2+2$ with bounds $x=1$ and $x=3$
Determine which is on top or whether they cross in the given interval


(they don't cross)
Set up the integral and evaluate:
$$\int_{1}^{3} x^2+2-\sqrt{x}\ \ dx=\int_{1}^{3} x^2+2-x^{1/2}\ \ dx$$
$$=\frac{x^3}{3}+2x-\frac{x^{3/2}}{3/2}\bigg|_{1}^{3}=\frac{x^3}{3}+2x-\frac{2\sqrt{x}^3}{3}\bigg|_{1}^{3}$$
$$\frac{3^3}{3}+2\cdot3-\frac{2\sqrt{3}^3}{3}-\left(\frac{1^3}{3}+2\cdot1-\frac{2\sqrt{1}^3}{3}\right)\approx 9.87$$

6. Find the area between the functions $f(x)=x^2-x-1$ and $g(x)=2x+3$ with bounds $x=-2$ and $x=2$

Determine which is on top or whether they cross in the given interval



They cross at x=-1 and x=4 (see problem 4 for details). x=-1 is in the interval, so we need two integrals:

$$\int_{-2}^{-1} x^2 - x - 1 - (2x+3)dx + \int_{-1}^{2} 2x + 3 - (x^2 - x - 1)dx$$

Integrate and evaluate each of the integrals:

$$\int_{-2}^{-1} x^2 - x - 1 - (2x+3)dx$$

$$= \int_{-2}^{-1} x^2 - x - 1 - 2x - 3dx$$

$$= \int_{-2}^{-1} x^2 - 3x - 4dx$$

$$= \frac{x^3}{3} - 3\cdot\frac{x^2}{2} - 4x\Big|_{-2}^{-1}$$

$$= \frac{(-1)^3}{3} - 3\cdot\frac{(-1)^2}{2} - 4\cdot(-1) - \left(\frac{(-2)^3}{3} - 3\cdot\frac{(-2)^2}{2} - 4\cdot(-2)\right)$$

$$= \frac{17}{6} \approx 2.8333$$

$$\int_{-1}^{2} (2x+3) - \left(x^2 - x - 1\right)dx$$

$$= \int_{-1}^{2} (2x+3) - x^2 + x + 1dx$$

$$= \int_{-1}^{2} -x^2 + 3x + 4dx$$

$$= -\frac{x^3}{3} + 3\cdot\frac{x^2}{2} + 4x\Big|_{-1}^{2}$$

$$= -\frac{(2)^3}{3} + 3\cdot\frac{(2)^2}{2} + 4\cdot(2) - \left(-\frac{(-1)^3}{3} + 3\cdot\frac{(-1)^2}{2} + 4\cdot(-1)\right)$$

$$= \frac{27}{2} = 13.5$$

Add the two areas together to get the total area

13.5+2.83333=16.3333 or $\frac{17}{6} + \frac{27}{2} = \frac{49}{3}$