4.1 Inverse functions:

Note: practice function compositions first!

A useful example: $f(x) = \frac{1}{x} + 2$ and $g(x) = \frac{1}{x-2}$

If we compose these two functions, and then we simplify, we get x

$$f \circ g(x) = f\left(\frac{1}{x-2}\right) = \frac{1}{\left(\frac{1}{x-2}\right)} + 2 = 1 \cdot \frac{x-2}{1} + 2 = x - 2 + 2 = x$$

And it works in both orders:

$$g \circ f(x) = g\left(\frac{1}{x} + 2\right) = \frac{1}{\left(\frac{1}{x} + 2\right) - 2} = \frac{1}{\frac{1}{x}} = 1 \cdot \frac{x}{1} = x$$

And we can look at this with numbers:
If we start with $x = 1$:

$$f(1) = \frac{1}{1} + 2 = 3$$

Then plug that number into *g*: $g(3) = \frac{1}{3-2} = \frac{1}{1} = 1$

If you start with 1, you end with 1. That's what an inverse function is….

- pick any number and plug it into f
- find the output
- plug that number into g
- The number you get back should be the number you started with!

Another notation for this is $f^{-1}(x) = g(x) = \frac{1}{x-2}$

You will need 2 new skills:

- finding an inverse function
- relating graphs of functions and their inverses

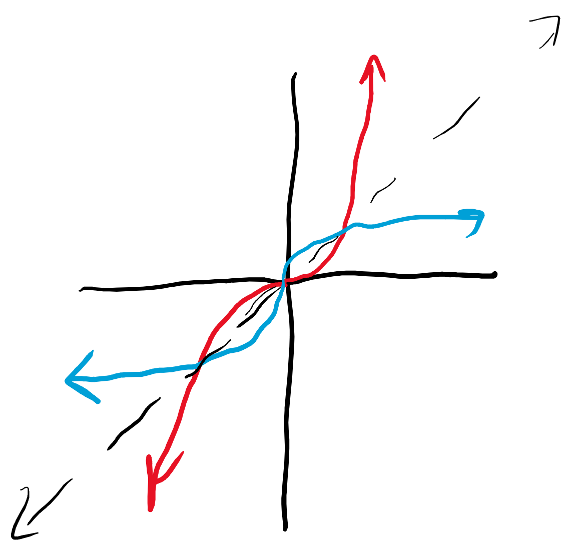
Example 2: Here is a pair of inverse functions

$f(x) = x^3$

x	y
-2	-8
-1	-1
-1/2	-1/8
0	0
1/2	1/8
1	1
2	8

$f^{-1}(x) = x^{1/3} = \sqrt[3]{x}$

x	y
-8	-2
-1	1
-1/8	-1/2
0	0
1/8	1/2
1	1
8	2



Note that the x and y coordinates of these functions and just switched, and the functions are reflections across the diagonal line $y = x$

How to find the inverse of a function

Steps:

1. Write the function as $y = ...$
2. Swap x and y in the equation
3. Solve for y. (this is the inverse function!)

Example 1: Easy

$f(x) = 3x + 5$

$y = 3x + 5$ (step 1)

$x = 3y + 5$ (step 2)

$x = 3y + 5$

$3y + 5 = x$

$3y = x - 5$

$y = \frac{1}{3}x - \frac{5}{3}$

so $f^{-1}(x) = \frac{1}{3}x - \frac{5}{3}$ (step 3)

Example 2: Harder: exponents

$f(x) = 2x^3 + 1$

$y = 2x^3 + 1$

$x = 2y^3 + 1$

$2y^3 + 1 = x$

$2y^3 = x - 1$ first undo/take care of things that are added or subtracted

$y^3 = \frac{x}{2} - \frac{1}{2}$ next undo/take care of things that are multiplied or divided

$y = \left(\frac{x}{2} - \frac{1}{2}\right)^{1/3}$ Finally, take care of exponents. Remember these exponent rules:

$\left(x^a\right)^b = x^{ab}$ and $x^{1/n} = \sqrt[n]{x}$ so you can do this:

$y^3 = x$ => $\left(y^3\right)^{1/3} = x^{1/3}$ => $y^{3\cdot(1/3)} = x^{1/3}$ => $y^1 = \sqrt[3]{x}$

$f^{-1}(x) = \sqrt[3]{\frac{x}{2} - \frac{1}{2}}$ or $f^{-1}(x) = \left(\frac{x}{2} - \frac{1}{2}\right)^{1/3}$

Example 3: Harder, fractions:

$f(x) = \frac{2}{x} + 4$

$y = \frac{2}{x} + 4$

$x = \frac{2}{y} + 4$

$y \cdot x = \left(\frac{2}{y} + 4\right) y$ You can multiply to get rid of the denominator. Be careful to multiply all terms by the same thing!

$xy = \frac{2}{y} \cdot y + 4y$

$xy = 2 + 4y$ Get all the y's on one side, and factor out if there is more than one term

$xy - 4y = 2$

$\frac{y(x-4)}{x-4} = \frac{2}{x-4}$ Then you can divide to finish solving for y

$y = \frac{2}{x-4}$

$f^{-1}(x) = \frac{2}{x-4}$

Example 4: fractions of the other sort

Starting from the other side:

$f(x) = \frac{3}{x-5}$

$y = \frac{3}{x-5}$

$x = \frac{3}{y-5}$

$(y-5) \cdot x = \frac{3}{y-5}(y-5)$ Multiply by the whole denominator to get rid of it (add parentheses)

$x(y-5) = 3$

$xy - 5x = 3$ Distribute and then isolate y by first adding/subtracting

$xy = 3 + 5x$

$\frac{xy}{x} = \frac{3}{x} + \frac{5x}{x}$ Then divide or multiply. Divide all of the terms.

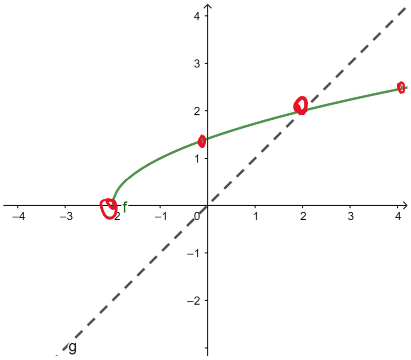
$y = \frac{3}{x} + 5$

$f^{-1}(x) = \frac{3}{x} + 5$

Remember that picture? If you ever need an inverse function, all you need to do is reflect across the diagonal line $y = x$. I suggest plotting a few specific points to help you out.

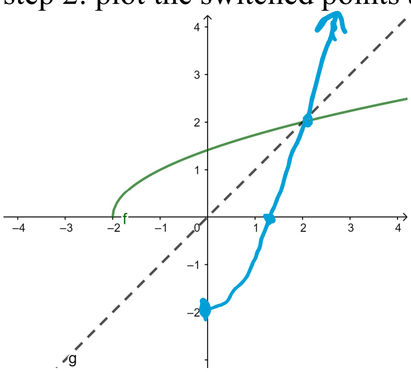
Example:

step 1: plot some points



(-2, 0)
(0, 1.2)
(2, 2)
(4, 2.5)

step 2: plot the switched points and connect for a reflected inverse function.



(0, 2)
(1.2, 0)
(2, 2)
(2.5, 4)

Homework solutions:

41. $f(x) = 2x + 4 \qquad g(x) = \frac{1}{2}x - 2$

$$f \circ g(x) = f\left(\frac{1}{2}x - 2\right) = 2\left(\frac{1}{2}x - 2\right) + 4$$

$$= 2 \cdot \frac{1}{2}x - 2 \cdot 2 + 4$$

$$= x - 4 + 4 = x$$

$$g \circ f(x) = g\left(2x + 4\right) = \frac{1}{2}(2x + 4) - 2$$

$$= \frac{1}{2} \cdot 2x + \frac{1}{2} \cdot 4 - 2$$

$$= x + 2 - 2 = x$$

59a. $y = 3x - 4$

$x = 3y - 4$

$3y - 4 = x$

$3y = x + 4$

$\frac{3y}{3} = \frac{x}{3} + \frac{4}{3}$

$f^{-1}(x) = \frac{x}{3} + \frac{4}{3}$

61a. $f(x) = x^3 + 1$

$y = x^3 + 1$

$x = y^3 + 1$

$y^3 + 1 = x$

$y^3 = x - 1$

$\left(y^3\right)^{1/3} = (x - 1)^{1/3}$

$y = (x - 1)^{1/3}$

$f^{-1}(x) = (x - 1)^{1/3}$ or $f^{-1}(x) = \sqrt[3]{x - 1}$

69a. $f(x)=\dfrac{1}{x-3}$

$y=\dfrac{1}{x-3}$

$x=\dfrac{1}{y-3}$

$(y-3)x=\dfrac{1}{y-3}(y-3)$

$xy-3x=1$

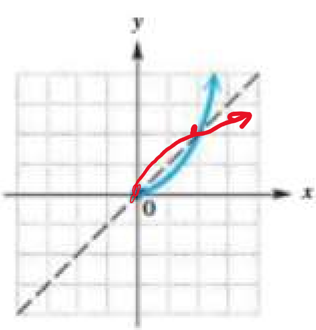
$xy=3x+1$

$\dfrac{xy}{x}=\dfrac{3x}{x}+\dfrac{1}{x}$

$y=3+\dfrac{1}{x}$

$f^{-1}=3+\dfrac{1}{x}$

77.



79.