Practice problems from Chapter 4 (about half of the final exam, maybe a little more):

1. For each of these relations, tell whether it is a function or not:

a. $\{(2,3),(2,4),(3,5)\}$ No, this is not a function, 2 is assigned to both 3 and 4 (one input 2 outputs)

b. $\{(2,3),(3,3),(4,3)\}$ Yes, Each input is assigned to only one output

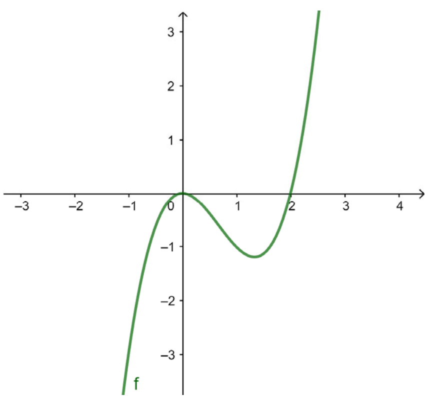
(is y a function of x: x is the input, y is the output)

c. $y = x^2$ Yes, if you know x, there is only one choice for y

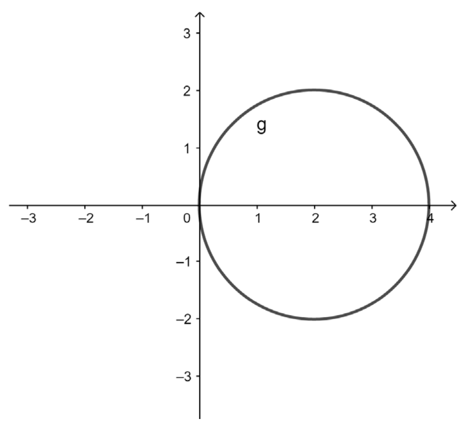
d. $x = y^2$ No, if x is a positive number (like 4) it could be assigned to either of two outputs (2 or -2)

e. $x = y^3$ Yes, if you know x, there is only one choice for y

f.



g.



Yes, for each x, there is only one point above it. No, if x=2, there are two points above it (y=2,y=-2)

2. For each of these pairs of functions, find $f \circ g(x)$. Simplify, and use the simplified form to tell whether the functions are inverses or not:

a. $f(x) = 3x + 5$ $g(x) = \frac{1}{3}x - 5$

$f \circ g(x) = f\left(\frac{1}{3}x - 5\right) = 3\left(\frac{1}{3}x - 5\right) + 5 = \frac{3}{3}x - 3 \cdot 5 + 5 = x - 10$. These functions are not inverses (you don't get back the same x you started with)

b. $f(x) = \frac{1}{x-2}$ $g(x) = \frac{1}{x} + 2$

$f \circ g(x) = f\left(\frac{1}{x} + 2\right) = \frac{1}{\left(\frac{1}{x} + 2\right) - 2} = \frac{1}{\frac{1}{x} + 2 - 2} = \frac{1}{\frac{1}{x}} = x$ Yes these functions are (probably) inverses (you get back the same x you started with. To prove for sure they are inverses, we should also check that $g \circ f(x) = x$, but that isn't part of what I'm asking you to do for this problem)

3. Find $f \circ g(x)$ for $f(x) = x^2 + 3x + 1$ $g(x) = x - 2$

$f \circ g(x) = f(x-2) = (x-2)^2 + 3(x-2) + 1$ (because the instructions don't say to simplify, you can stop here)

4. Find the inverse function for each of these functions:

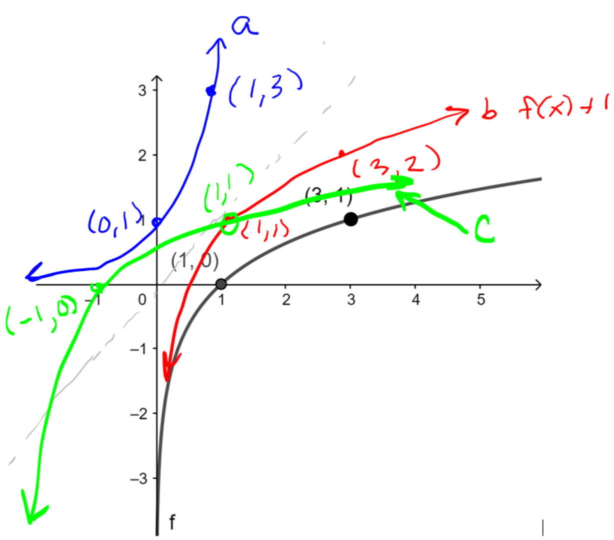
a. $f(x)=3x+4$ $y=3x+4$ $x=3y+4$ $x-4=3y$ $\frac{x-4}{3}=y$ $f^{-1}(x)=\frac{x-4}{3}$	b. $f(x)=2x^3+1$ $y=2x^3+1$ $x=2y^3+1$ $x-1=2y^3$ $\frac{x-1}{2}=y^3$ $\sqrt[3]{\frac{x-1}{2}}=y$ $f^{-1}(x)=\sqrt[3]{\frac{x-1}{2}}$	c. $f(x)=3\cdot 2^x$ $y=3\cdot 2^x$ $x=3\cdot 2^y$ $\frac{x}{3}=2^y$ $y=\log_2\left(\frac{x}{3}\right)$ $f^{-1}(x)=\log_2\left(\frac{x}{3}\right)$ OR $y=3\cdot 2^x$ $x=3\cdot 2^y$ $\frac{x}{3}=2^y$ $\log\left(\frac{x}{3}\right)=\log\left(2^y\right)$ $\log\left(\frac{x}{3}\right)=y\log\left(2\right)$ $\frac{\log(x/3)}{\log(2)}=y$ $f^{-1}(x)=\frac{\log(x/3)}{\log(2)}$ These two different-looking answers are equal because $\log_2(x)=\frac{\log(x)}{\log(2)}$

Note: if you forget to write the last line with the f^{-1} notation, I won't deduct points for that.

5. If the graph of $y=f(x)$ given below, graph:

a. $y=f^{-1}(x)$ b. $y=f(x)+1$ c. $y=f(x+2)$

(show exactly what happens to the two points labelled)



The interest rate equations are: $A = P\left(1+\frac{r}{n}\right)^{nt}$ for interest compounded n times per year and $A = P\mathrm{e}^{rt}$ for interest compounded continuously.

6. If an amount of $3,000 is invested at an annual interest rate of 2.8% compounded quarterly, what is the value of the investment after 5 years?

$$A = (3,000)\left(1+\frac{.028}{4}\right)^{4\cdot 5} = 3,000(1.007)^{20} \approx \$3449.14$$

7. If the same amount ($3,000) is invested at the same annual interest rate (2.8%), but the interest is compounded continuously, what is the value of the investment after 5 years?

$$A = 3,000\cdot \mathrm{e}^{(0.028)\cdot 5} = 3,000\cdot e^{0.14} \approx \$3450.82$$

8. How much money must be invested at an annual interest rate of 2.8%, compounded quarterly, so that the investment will yield $6000 in 8 years?

$$6,000 = P\left(1+\frac{.028}{4}\right)^{4\cdot 8}$$

$$6,000 = P(1.007)^{32}$$

$$P = \frac{6,000}{(1.007)^{32}} \approx \$4799.63$$

9. How much money must be invested at an annual interest rate of 2.8%, compounded continuously, so that the investment will yield $6000 in 8 years?

$$6,000 = P\mathrm{e}^{0.028\cdot 8}$$

$$6,000 = Pe^{0.224}$$

$$P = \frac{6,000}{e^{0.224}} \approx \$4795.89$$

10. If an amount of money is invested at an annual interest rate of 2.8%, compounded <u>quarterly</u>, how long will it take for the investment to double in value?

For this solution, I will pretend that I am investing \$1,000, so double that would be \$2,000 $2,000 = 1,000\left(1+\frac{.028}{4}\right)^{4\cdot t}$ $2,000 = 1,000(1.007)^{4t}$ $\frac{2,000}{1,000} = (1.007)^{4t}$ $2 = (1.007)^{4t}$ $\log(2) = \log\left(1.007^{4t}\right)$ $\log 2 = 4t\log(1.007)$ $4t = \frac{\log 2}{\log(1.007)} \approx 99.3672$ $t \approx \frac{99.3672}{4} \approx 24.84\ years$	I could also use the symbol P for the investment, and double that would be 2P, so the first half of the problem would look like this: $2P = P\left(1+\frac{.028}{4}\right)^{4\cdot t}$ $2P = P(1.007)^{4t}$ $\frac{2P}{P} = (1.007)^{4t}$ $2 = (1.007)^{4t}$	If you know the change of base formula, so you can evaluate things like $\log_2 5 = \frac{\log 5}{\log 2}$ Then you can do the second half of the problem in this way (by changing the exponent equation into a log equation) $2 = (1.007)^{4t}$ $\log_{1.007}(2) = 4t$ $\frac{\log 2}{\log(1.007)} = 4t$ $99.3672 \approx 4t$ $\frac{99.3672}{4} \approx t$ $t \approx 24.84\ years$

11. If an amount of money is invested at an annual interest rate of 2.8%, compounded <u>continuously</u>, how long will it take for the investment to double in value?

For this solution, I will pretend that I am investing \$1,000, so double that would be \$2,000 $2,000 = 1,000e^{0.028t}$ $\frac{2,000}{1,000} = e^{0.028t}$ $2 = e^{0.028t}$ $\ln(2) = \ln\left(e^{0.028t}\right)$ $\ln 2 = 0.028t\ln(e)$ $\ln 2 = 0.028t$ $t = \frac{\ln(2)}{0.028} \approx 24.75\ years$	I could also use the symbol P for the investment, and double that would be 2P, so the first half of the problem would look like this: $2P = Pe^{0.028t}$ $\frac{2P}{P} = e^{0.028t}$ $2 = e^{0.028t}$	This is one where it's really easy to do the second half by changing into a logarithm: $2 = e^{0.028t}$ $\log_e(2) = 0.028t$ $\ln(2) = 0.028t$ $\frac{\ln 2}{0.028} = t$ $t \approx 24.75\ years$

12. Plutonium-241 decays according to the function $A(t) = A_0 e^{-0.053t}$. How long will it take some amount of Plutonium to decay to half of its amount?

The same 3 variations on the solution strategy:

Pick an easy number for the starting amount of plutonium-241. I'm going to pick 4g. $2 = 4e^{-0.053t}$ $\frac{2}{4} = e^{-0.053t}$ $0.5 = e^{-0.053t}$ $\ln(0.5) = \ln\left(e^{-0.053t}\right)$ $\ln(0.5) = -0.053t\ln(e)$ $\ln(0.5) = -0.053t$ $t = \frac{\ln(0.5)}{-0.053} \approx 13.1\, years$	I could also use the symbol A_0 for the initial amount, and half of that would be $A_0 / 2$, so the first half of the problem would look like this: $A_0 / 2 = A_0 e^{-0.053t}$ $\frac{A_0 / 2}{A_0} = e^{-0.053t}$ $1/2 = e^{-0.053t}$ $0.5 = e^{-0.053t}$	This is another one where it's really easy to do the second half by changing into a logarithm $0.5 = e^{-0.053t}$ $\log_e(0.5) = -0.053t$ $\ln(0.5) = -0.053t$ $t = \frac{\ln(0.5)}{-0.053} \approx 13.1\, years$

13. Expand these logarithmic expressions:

a. $\log_3\left(\frac{3\sqrt{x}}{y^2}\right) = \log_3(3) + \log_3\sqrt{x} - \log_3 y^2 = 1 + \log_3 x^{1/2} - \log_3 y^2 = 1 + \frac{1}{2}\log_3 x - 2\log_3 y$

b. $\log\left(\frac{x}{\sqrt{yz}}\right) = \log\left(\frac{x}{(yz)^{1/2}}\right) = \log\left(\frac{x}{y^{1/2}z^{1/2}}\right) = \log x - \log y^{1/2} - \log z^{1/2} = \log x - \frac{1}{2}\log y - \frac{1}{2}\log z$

c. $\ln\left(3\sqrt[4]{xy}\right) = \ln\left(3(xy)^{1/4}\right) = \ln\left(3x^{1/4}y^{1/4}\right) = \ln 3 + \ln x^{1/4} + \ln y^{1/4} = \ln 3 + \frac{1}{4}\ln x + \frac{1}{4}\ln y$

14. Combine these logarithmic expressions:

a. $\log x + 2\log y - 3\log z - \log w = \log x + \log y^2 - \log z^3 - \log w = \log\frac{xy^2}{z^3 w}$

b. $\ln(3) + 2\ln x - \ln(x+1) = \ln 3 + \ln x^2 - \ln(x+1) = \ln\frac{3x^2}{x+1}$

c. $\log_2 x + 2\log_2(x+1) = \log_2 x + \log_2(x+1)^2 = \log_2\left(x(x+1)^2\right)$

15. Solve the exponential equations:

a. $2^{(x-1)} = 8$ Solution if you notice that $8 = 2^3$ $2^{(x-1)} = 2^3$ $x - 1 = 3$ $x = 4$	A solution using log: $2^{(x-1)} = 8$ $\log(2^{x-1}) = \log 8$ $(x-1)\log(2) = \log 8$ $\frac{(x-1)\log(2)}{\log(2)} = \frac{\log 8}{\log(2)}$ $x - 1 = \frac{\log 8}{\log 2}$ $x - 1 \approx 3$ $x \approx 3 + 1$ $x \approx 4$	A solution using ln: $2^{(x-1)} = 8$ $\ln(2^{x-1}) = \ln 8$ $(x-1)\ln(2) = \ln 8$ $\frac{(x-1)\ln(2)}{\ln(2)} = \frac{\ln 8}{\ln(2)}$ $x - 1 = \frac{\ln 8}{\ln(2)}$ $x - 1 \approx 3$ $x \approx 3 + 1$ $x \approx 4$	A solution using $\log_2$ $2^{(x-1)} = 8$ $\log_2(8) = x - 1$ $3 = x - 1$ $4 = x$ Note: if you don't notice that $8 = 2^3$, you have to do $\log_2(8) = \frac{\log 8}{\log 2}$ or $\log_2(8) = \frac{\ln 8}{\ln 2}$
b. $3(2^{x+1}) = 12$ Solution if you notice that $4 = 2^2$ $\frac{3(2^{x+1})}{3} = \frac{12}{3}$ $2^{x+1} = 4$ $2^{x+1} = 2^2$ $x + 1 = 2$ $x = 1$	A solution using log: $\frac{3(2^{x+1})}{3} = \frac{12}{3}$ $2^{x+1} = 4$ $\log(2^{x+1}) = \log(4)$ $(x+1)\log 2 = \log 4$ $x + 1 = \frac{\log 4}{\log 2}$ $x + 1 \approx 2$ $x \approx 1$	A solution using ln: $\frac{3(2^{x+1})}{3} = \frac{12}{3}$ $2^{x+1} = 4$ $\ln(2^{x+1}) = \ln(4)$ $(x+1)\ln 2 = \ln 4$ $x + 1 = \frac{\ln 4}{\ln 2}$ $x + 1 \approx 2$ $x \approx 1$	A solution using $\log_2$ $\frac{3(2^{x+1})}{3} = \frac{12}{3}$ $2^{x+1} = 4$ $\log_2 4 = x + 1$ $2 = x + 1$ $1 = x$ Note: if you don't notice that $4 = 2^2$, you have to do $\log_2(4) = \frac{\log 4}{\log 2}$ or $\log_2(4) = \frac{\ln 4}{\ln 2}$
c. $5(1.2)^{x-4} = 14$ A solution using log: $\frac{5(1.2)^{x-4}}{5} = \frac{14}{5}$ $1.2^{x-4} = 2.8$ $\log(1.2^{x-4}) = \log(2.8)$ $(x-4)\log(1.2) = \log(2.8)$ $x - 4 = \frac{\log 2.8}{\log 1.2}$ $x - 4 \approx 5.6473$ $x = 9.6473$	A solution using ln: $\frac{5(1.2)^{x-4}}{5} = \frac{14}{5}$ $1.2^{x-4} = 2.8$ $\ln(1.2^{x-4}) = \ln(2.8)$ $(x-4)\ln(1.2) = \ln(2.8)$ $x - 4 = \frac{\ln 2.8}{\ln 1.2}$ $x - 4 \approx 5.6473$ $x = 9.6473$	A solution using $\log_{1.2}$ $\frac{5(1.2)^{x-4}}{5} = \frac{14}{5}$ $1.2^{x-4} = 2.8$ $\ln_{1.2}(2.8) = x - 4$ $\frac{\log 2.8}{\log 1.2} = x - 4$ $5.6473 \approx x - 4$ $9.6473 \approx x$	

d. $4\cdot 2^{3x}=18$ A solution using log: $\frac{4\cdot 2^{3x}}{4}=\frac{18}{4}$ $2^{3x}=4.5$ $\log 2^{3x}=\log(4.5)$ $3x\log 2=\log 4.5$ $3x=\frac{\log 4.5}{\log 2}$ $3x\approx 2.1699$ $x\approx\frac{2.1699}{3}$ $x\approx .7233$	A solution using ln: $\frac{4\cdot 2^{3x}}{4}=\frac{18}{4}$ $2^{3x}=4.5$ $\ln 2^{3x}=\ln(4.5)$ $3x\ln 2=\ln 4.5$ $3x=\frac{\ln 4.5}{\ln 2}$ $3x\approx 2.1699$ $x\approx\frac{2.1699}{3}$ $x\approx .7233$	A solution using $\log_2$ $\frac{4\cdot 2^{3x}}{4}=\frac{18}{4}$ $2^{3x}=4.5$ $\log_2(4.5)=3x$ $\frac{\log 4.5}{\log 2}=3x$ $2.1699\approx 3x$ $\frac{2.1699}{3}\approx 3$ $0.7233\approx 3$	

16. Solve the logarithmic equations:

a. $\log_3(2x+1)-\log_3(x-3)=2$ $\log_3\frac{(2x+1)}{x-3}=2$ $\frac{2x+1}{x-3}=3^2$ $\frac{2x+1}{x-3}=9$ $2x+1=9(x-3)$ $2x+1=9x-27$ $28=7x$ $x=4$	b. $\ln(x+1)+\ln(x+2)-\ln(4)=\ln(3)$ $\ln\left(\frac{(x+1)(x+2)}{4}\right)=\ln(3)$ $\frac{(x+1)(x+2)}{4}=3$ $(x+1)(x+2)=12$ $x^2+2x+x+2=12$ $x^2+3x-10=0$ $(x+5)(x-2)=0$ $x=-5,\ x=2$ If you plug these back in, you'll find that only one works and makes sense (the other has an undefined log), so really the answer is just x=2
c. $\log_2(2x+5)-\log_2(x-1)=3$ $\log_2\left(\frac{2x+5}{x-1}\right)=3$ $\frac{2x+5}{x-1}=2^3$ $\frac{2x+5}{x-1}=8$ $2x+5=8(x-1)$ $2x+5=8x-8$ $13=6x$ $x=13/6$	d. $\log(x+1)+\log(x-2)=1$ $\log\left((x+1)(x-2)\right)=1$ $(x+1)(x-2)=10^1$ $x^2-2x+x-2=10$ $x^2-x-12=0$ $(x-4)(x+3)=0$ $x=4,\ x=-3$ If you plug these back in, you'll find that only one works and makes sense (the other has an undefined log), so really the answer is just x=4

16. e. $\log(x+1)-\log(x-2)=\log(4)$

$$\log\left(\frac{x+1}{x-2}\right)=\log(4)$$
$$\frac{x+1}{x-2}=4$$
$$x+1=4(x-2)$$
$$x+1=4x-8$$
$$9=3x$$
$$3=x$$

17. Solve by factoring: $2x^2-5x-12=0$

$$2x^2-5x-12=0$$
$$(2x+3)(x-4)=0$$
$$x=-3/2, \quad x=4$$

18. Solve and leave the answer in exact simplified form (square roots, fractions, but no decimals)

a. $2x^2-2x+3=0$ $x=\frac{-(-2)\pm\sqrt{(-2)^2-4\cdot2\cdot3}}{2\cdot2}$ $=\frac{2\pm\sqrt{4-24}}{4}$ $=\frac{2\pm\sqrt{-20}}{4}$ $=\frac{2\pm\sqrt{-1\cdot4\cdot5}}{4}$ $=\frac{2\pm2i\sqrt{5}}{4}$ $=\frac{2}{4}\pm\frac{2i\sqrt{5}}{4}$ $=\frac{1}{2}\pm\frac{i\sqrt{5}}{2}$	b. $x^2+4x-8=0$ $x=\frac{-(4)\pm\sqrt{(4)^2-4\cdot1\cdot(-8)}}{2\cdot1}$ $=\frac{-4\pm\sqrt{16+32}}{2}$ $=\frac{4\pm\sqrt{48}}{2}$ $=\frac{4\pm\sqrt{4\cdot4\cdot3}}{2}$ $=\frac{4\pm2\cdot2\sqrt{3}}{2}$ $=\frac{4}{2}\pm\frac{4\sqrt{3}}{2}$ $=2\pm2\sqrt{3}$

19. Factor each expression:

a. $x^2+4x=x(x+4)$

b. $x^3-10x^2+24x=x(x^2-10x+24)=x(x-6)(x-4)$

20. Solve each equation:

a. $\frac{x+1}{x^2-9}-\frac{x+3}{x^2+5x+6}=\frac{4}{x^2-x-6}$

$$\frac{x+1}{(x-3)(x+3)}-\frac{x+3}{(x+3)(x+2)}=\frac{4}{(x-3)(x+2)}$$

$$\frac{x+1}{(x-3)(x+3)}\cdot\frac{(x+2)(x+3)(x-3)}{1}-\frac{x+3}{(x+3)(x+2)}\cdot\frac{(x+2)(x+3)(x-3)}{1}=\frac{4}{(x-3)(x+2)}\cdot\frac{(x+2)(x+3)(x-3)}{1}$$

$$\frac{x+1}{\cancel{(x-3)(x+3)}}\cdot\frac{(x+2)\cancel{(x+3)(x-3)}}{1}-\frac{x+3}{\cancel{(x+3)(x+2)}}\cdot\frac{\cancel{(x+2)(x+3)}(x-3)}{1}=\frac{4}{\cancel{(x-3)}\cancel{(x+2)}}\cdot\frac{\cancel{(x+2)}(x+3)\cancel{(x-3)}}{1}$$

$$(x+1)(x+2)-(x+3)(x-3)=4(x+3)$$

$$x^2+2x+x+2-(x^2-3x+3x-9)=4x+12$$

$$x^2+3x+2-(x^2-9)=4x+12$$

$$x^2+3x+2-x^2+9=4x+12$$

$$3x+11=4x+12$$

$$-1=x$$

b. $(x+3)^{2/3}=36$

$$\left((x+3)^{2/3}\right)^{3/2}=36^{3/2}$$

$$(x+3)^1=\left(36^{1/2}\right)^3$$

$$x+3=\sqrt{36}^3$$

$$x+3=6^3$$

$$x+3=216$$

$$x=213$$

21. An ideal gas satisfies the equation PV=nRT, where P is the pressure in atm, V is the volume in Liters, T is the temperature in degrees kelvin, n is the number of moles, and R is a constant.

a. Solve for the constant R.

$$\frac{PV}{nT}=\frac{nRT}{nT}$$

$$R=\frac{PV}{nT}$$

b. Given n=2.5 mol of a gas, when the temperature is 275 K and the pressure is .95 atm, then the volume is 59 L. If the air pressure stays the same (.95 atm), and the amount of gas stays the same (2.5 mol) and the temperature increases to 310 K, what is the new volume?

$R = \frac{.95 \cdot 59}{2.5 \cdot 275} = \frac{56.05}{687.4} \approx .0815$ and $R = \frac{.95 \cdot V}{2.5 \cdot 310} = \frac{.95V}{775}$

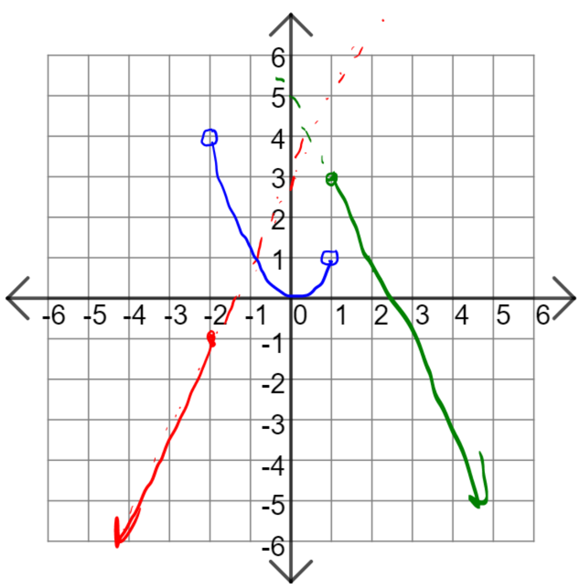
$\frac{.95V}{775} \approx .0815$

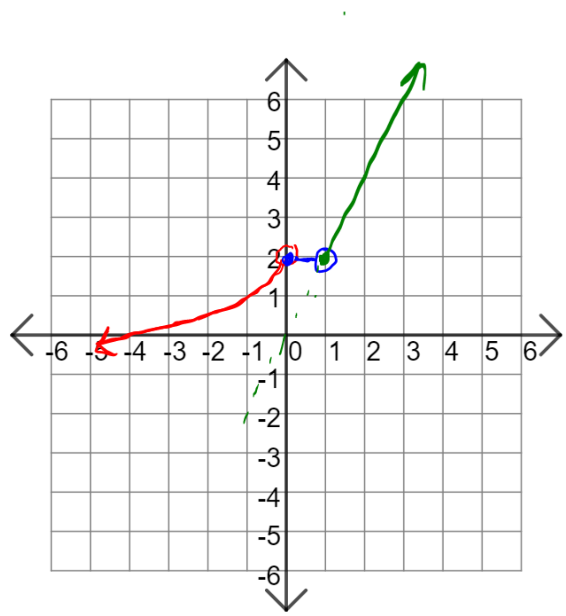
So $.95V \approx .0815 \cdot 775 \approx 63.1625$

$V \approx \frac{63.1625}{.95} \approx 66.5\,L$

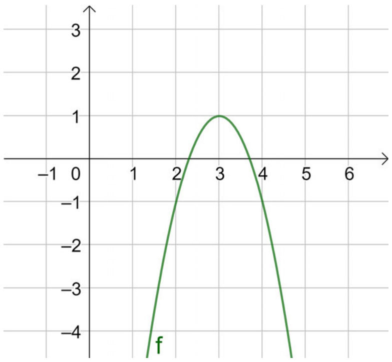
There will be some problems like these from Chapter 2:

22. Graph the functions a. $y = \begin{cases} 2x+3 & \text{if} \quad x \le -2 \\ x^2 & \text{if} \quad -2 < x < 1 \\ -2x+5 & \text{if} \quad 1 \le x \end{cases}$ b. $y = \begin{cases} \sqrt{-x}+2 & \text{if} \quad x < 0 \\ 2 & \text{if} \quad 0 \le x < 1 \\ 2x & \text{if} \quad 1 \le x \end{cases}$

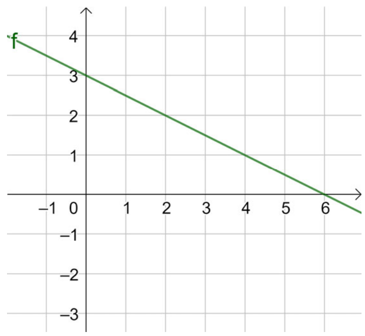




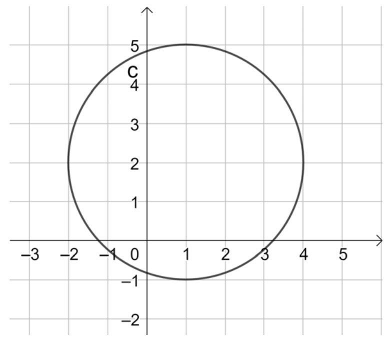
23. Write the equation of each of these functions or relations:



$$y = -2(x-3)^2 + 1$$



$$y = -\frac{1}{2}x + 3$$



$$(x-1)^2 + (y-2)^2 = 9$$

24. Write an equation of a line through points (2,3) and (5,1)

$$m = \frac{1-3}{5-2} = \frac{-2}{3}$$

$$y - 3 = \frac{-2}{3}(x-2)$$

$$y - 3 = -\frac{2}{3}x + \frac{4}{3}$$

$$y = -\frac{2}{3}x + \frac{4}{3} + \frac{9}{3}$$

$$y = -\frac{2}{3}x + \frac{13}{3}$$

Other correct form of the answer is

$$3y = -2x + 13$$

$$2x + 3y = 13$$

There will be a factorization problem like one of these from Chapter 3:

25. Completely factor each of these polynomials:

a. $f(x) = 2x^3 + 13x^2 + 17x - 12$ given that (x+4) is a factor

Factor out x+4 by synthetic or long division:

$$\begin{array}{r|rrrr} -4 & 2 & 13 & 17 & -12 \\ & & -8 & -20 & 12 \\ \hline & 2 & 5 & -3 & 0 \end{array}$$

So $f(x) = (x+4)(2x^2 + 5x - 3)$

And then factor the quadratic part:

$$\begin{aligned} f(x) &= (x+4)(2x^2 + 5x - 3) \\ &= (x+4)(2x-1)(x+3) \end{aligned}$$

b. $f(x) = 3x^3 + 4x^2 - 17x - 6$ given that 2 is a zero.

Factor out x-2 by synthetic or long division:

$$\begin{array}{r|rrrr} 2 & 3 & 4 & -17 & -6 \\ & & 6 & 20 & 6 \\ \hline & 3 & 10 & 3 & 0 \end{array}$$

$f(x) = (x-2)(3x^2 + 10x + 3)$

And then factor the quadratic part:

$f(x) = (x-2)(3x+1)(x+3)$